JHEP HYPER VERSION

Received: *December 17, 1999*, Accepted: *January 7, 2000*

Nonlinear instantons from supersymmetric p-branes

Marcos Mariño[*], **Ruben Minasian**[†], **and Gregory Moore**[*]

Department of Physics, Yale University
New Haven, CT 06520, USA
E-mail: , marcos.marino@yale.edu, ruben.minasian@yale.edu, moore@castalia.physics.yale.edu

Andrew Strominger

Jefferson Physical Laboratories, Harvard University
Cambridge, MA 02138, USA
E-mail: andy@pauli.harvard.edu

Abstract: Supersymmetric configurations of type-II D-branes with nonzero gauge field strengths in general supersymmetric backgrounds with nonzero B fields are analyzed using the κ-symmetric worldvolume action. It is found in dimension four or greater that the usual instanton equation for the gauge field obtains a nonlinear deformation. The deformation is parameterized by the topological data of the B-field, the background geometry and the cycle wrapped by the brane. In the appropriate dimensions, limits and settings these equations reduce to deformed instanton equations recently found in the context of noncommutative geometry as well as those following from lagrangeans based on Bott-Chern forms. We further consider instantons comprised of M5-branes wrapping a Calabi-Yau space with non-vanishing three-form field strengths. It is shown that the instanton equations for the three-form are related to the Kodaira-Spencer equations.

Keywords: Field Theories in Higher Dimensions, Nonperturbative Effects, p-branes, D-branes.

[*]Address after January 1, 2000: Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA

[†]Centre de Physique Théorique, Ecole Polytechnique, F-91128 Palaiseau, France

Contents

1. Introduction and conclusion

Supersymmetric cycle conditions are of importance in string theory because these are the equations governing the existence of BPS states and supersymmetric instantons associated with wrapped p-branes. These supersymmetric cycle conditions are equations for a *pair* consisting of an embedded manifold in spacetime, together with a gauge connection on the embedded manifold. There is by now a rich literature for the special case in which the gauge field strength F on the brane as well as the background supergravity potentials B vanish. The general case of nonzero F or B in contrast has received less attention.

In this paper the more general case is addressed. D-branes with nonzero U(1) gauge fields F in supersymmetric IIA and IIB supergravity backgrounds with nonzero B fields are analyzed using the recently discovered κ-symmetric worldvolume actions [2, 3, 4]. In all cases considered, we find that the conditions governing supersymmetrically embedded cycles are unchanged by $B \neq 0$.[1] In contrast, the equation for the gauge field on the cycle is, in dimension four or greater, an intriguing nonlinear deformation of the usual instanton equation. The deformed equation depends on the topological data of both the cycle and the B field.

An important motivation for this work is its application to the study of Yang-Mills theory in a non-commutative geometry. The connection between noncommutative geometry and string theory was first noted by Connes, Douglas, and Schwarz [5]. Recently, Seiberg and Witten [6] have related the study of BPS configurations of D-branes to various aspects of noncommutative geometry. In particular it is shown how the presence of a nonzero B-field deforms the instanton equations, in a way related to the noncommutative instanton equations of Nekrasov and Schwarz [7]. We will find that our deformed equations reduce to those of [6] in the appropriate limit and setting. It further generalizes those equations to curved backgrounds and to higher dimensions. In $\mathbb{R}^4$ we find an additional deformation parameter away from the limit considered in [6].[2] In particular we find that the hermitean Yang-Mills equations of six dimensions are also modified, and that the discussion of [6] relating BPS conditions to the noncommutative instanton equations of Nekrasov-Schwarz [7] generalizes nicely to this case. We also find that the G_2 and Spin(7) instanton equations are not deformed, and the instanton equations on calibrated 4-cycles in such manifolds are likewise not deformed.

This paper also has two unexpected results. The first novelty is that the nonlinear deformations of the instanton equations arising from BPS conditions are closely related to the equations for hermitean metrics on holomorphic vector bundles discussed by Leung [8] and by Losev et al. [9]. The second novelty appears when we further extend our analysis to the case of an M5-brane wrapping a Calabi-Yau threefold. The M5-brane worldvolume contains a closed rank three antisymmetric tensor field strength H which obeys a nonlinear "self-duality constraint" [10, 11]. In [12] a nonlinear change of variables to a 3-form h was found such that h is self-dual, but satisfies a nonlinear equation of motion. We show that, when combined with the condition of preserving a supersymmetry, this nonlinear equation of motion is just the Kodaira-Spencer equation together with a gauge condition. The gauge condition is a deformation of the standard one, and it shown that a solution exists through third order in perturbation theory.

This paper raises a number of interesting open problems. In principle nonabelian versions of our deformed equations could be derived via string theory by

[1]The cases we consider do not include certain interesting singularities, such as occur in BIons.
[2]This additional parameter was known to the authors of [6].

considering multiply-wrapped D-branes. In practice this would be difficult. However in most cases the equations we write have obvious non-abelian generalizations which we expect to apply to this case. Moreover, the relation to [9] opens up a host of new issues and questions related to the possible use of higher dimensional "*bc* systems."

A brief outline of the paper is the following. Section 2 is a lightning review of the relevant κ-symmetric D-branes [2, 3]. In section 3 we show that the cycle embeddings are undeformed and derive the deformed instanton equations for a variety of branes embedded in manifolds of SU(2), SU(3), G_2, SU(4) and Spin(7) holonomy. A group theoretic analysis of the equations and their solutions for $\mathbb{R}^4$, $\mathbb{R}^6$ and $\mathbb{R}^8$ is given in section 4. In section 5 we relate a limit of the $\mathbb{R}^4$ equations to those derived by Seiberg and Witten in the context of noncommutative geometry, and then generalize the relation to $\mathbb{R}^6$. In section 6, we briefly discuss the relation of the present results to those of Leung [8] and Losev et. al. [9]. Finally, in section 7, we analyze the supersymmetry conditions for an M5 brane wrapping a Calabi-Yau threefold.

2. Review of κ-symmetry and the Γ-operator

We start with a very brief review of D-brane actions, κ-symmetry and the properties of the Γ-operator that are crucial for our discussion. The action was constructed in [2, 3] and is of the form (for constant dilaton)

$$I_p = I_{\mathrm{DBI}} + I_{\mathrm{WZ}} = -T_p \int_W d^{p+1}\sigma \sqrt{\det(g_{\mu\nu} + M_{\mu\nu})} + \mu \int_W C \wedge e^M . \tag{2.1}$$

Here W is the brane worldvolume. T_p and μ are the brane tensions and charges, respectively. The metric on W

$$g_{\mu\nu} = E_\mu{}^a E_\nu{}^b \eta_{ab} , \tag{2.2}$$

is induced by supermaps $Z = (X, \theta)$ and super-vielbein

$$E_\mu{}^{\mathbf{A}} = \partial_\mu Z^{\mathbf{M}} E_{\mathbf{M}}{}^{\mathbf{A}} . \tag{2.3}$$

The frame index $\mathbf{A}$ decomposes into 10 vector (a) and 32 spinor (α) indices (and an $\mathrm{SL}(2,\mathbb{R})$ index (A) in type-IIb case). $M_{\mu\nu}$ $(\mu = 1, \ldots, (p+1))$ is the modified 2-form field strength $M = 2\pi\alpha'(F + B)$ with B being the pull-back of the NS two-form field to the worldvolume. The couplings to the background RR fields are given by the second term in (2.1), where

$$C = \sum_{r=0}^{10} C^{(r)}$$

is a formal sum of the RR fields $C^{(r)}$ (we are ignoring the gravitational couplings here).

Since in addition to κ-symmetry, the classical D-brane actions have spacetime supersymmetry, we can combine both, and in particular determine the fraction of unbroken supersymmetry by the dimension of the solution space of the equation [1, 13]:

$$(1-\Gamma)\eta = 0\,, \tag{2.4}$$

where η is the spacetime supersymmetry parameter, and Γ is an hermitean traceless matrix:

$$\operatorname{tr}\Gamma = 0\,, \qquad \Gamma^2 = 1\,. \tag{2.5}$$

The explicit form for Γ will be important for our analysis (note that we are working with euclidean branes):

$$\Gamma = \frac{\sqrt{|g|}}{\sqrt{|g+M|}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} \gamma^{\mu_1\nu_1\cdots\mu_n\nu_n} M_{\mu_1\nu_1}\cdots M_{\mu_n\nu_n} J^{(n)}_{(p)}\,, \tag{2.6}$$

where $|g| := \det(g_{\mu\nu})$, $|g+M| := \det(g_{\mu\nu}+M_{\mu\nu})$, and

$$J^{(n)}_{(p)} = \begin{cases} i(\Gamma_{11})^{n+\frac{p-2}{2}}\Gamma_{(0)} & \text{for IIA,} \\ (-1)^n(\sigma_3)^{n+\frac{p-3}{2}}\sigma_2\otimes\Gamma_{(0)} & \text{for IIB.} \end{cases} \tag{2.7}$$

Finally, $\Gamma_{(0)}$ is defined as

$$\Gamma_{(0)} = \frac{1}{(p+1)!\sqrt{|g|}} \epsilon^{\mu_1\ldots\mu_{(p+1)}} \gamma_{\mu_1\ldots\mu_{(p+1)}}\,, \tag{2.8}$$

with

$$\gamma_\mu = E_\mu{}^a \Gamma_a\,,$$

where $\{\Gamma_a; a = 0,\ldots,9\}$ are the spacetime gamma-matrices.

A very important feature is that the non-linear dependence on M can be expressed in the form [4]:

$$\Gamma = e^{-a/2}\,\Gamma'_{(0)}\,e^{a/2}\,, \tag{2.9}$$

where $a = a(M)$ contains all the dependence on M and $\Gamma'_{(0)}$ (which depends only on X) is also an hermitean traceless matrix (i.e. tr $\Gamma'_{(0)} = 0$ and $(\Gamma'_{(0)})^2 = 1$) given by:

$$\Gamma'_{(0)} = \begin{cases} (\Gamma_{11})^{\frac{p-2}{2}}\Gamma_{(0)} & \text{for IIA,} \\ (\sigma_3)^{\frac{p-3}{2}}\sigma_2\otimes\Gamma_{(0)} & \text{for IIB.} \end{cases} \tag{2.10}$$

An explicit expression for a can be found in a local euclidean frame in which M is skew-diagonal [4]:

$$a = \begin{cases} -\dfrac{1}{2}Y_{jk}\gamma^{jk}\Gamma_{11} & \text{for IIA,} \\ \dfrac{1}{2}Y_{jk}\sigma_3\otimes\gamma^{jk} & \text{for IIB,} \end{cases} \tag{2.11}$$

$p+1$	SU(2)	SU(3)	G_2	SU(4)	Spin(7)
2	divisor/SLag	holomorphic	—	holomorphic	—
3	—	SLag	associative	—	—
4	X	divisor	coassociative	Cayley	Cayley
5	—	—	—	—	—
6	—	X	—	divisor	—
7	—	—	X	—	—
8	—	—	—	X	X

Table 1: Cycles in manifolds of irreducible non-trivial holonomy.

where Y is

$$Y = \frac{1}{2} Y_{ik} e^i \wedge e^k = \sum_{r=1}^{[(p+1)/2]} \phi_{2r-1,2r} e^{2r-1} \wedge e^{2r} \tag{2.12}$$

and is related to M by

$$M = \frac{1}{2} M_{ik} e^i \wedge e^k = \sum_{r=1}^{[(p+1)/2]} \tan \phi_{2r-1,2r} e^{2r-1} \wedge e^{2r} \,. \tag{2.13}$$

In these equations, e is the vielbein on the worldvolume ($g_{\mu\nu} = e^i{}_\mu e^k{}_\nu \eta_{ik}$, with $i, k = 0, \ldots, p$). Notice that, in any orthonormal frame, if we define the matrices $M = \sum_{i,j} M_{ij} T_{ij}$, $Y = \sum_{i,j} Y_{ij} T_{ij}$, where $T_{ij} = e_{ij} - e_{ji}$ and e_{ij} are matrix units, then the relation between M and Y is

$$M = \tanh(Y) \,. \tag{2.14}$$

As a final remark about notation, we will use $\mathrm{vol}_{p+1} = \sqrt{|g|} d^{p+1}\xi$ to denote the canonical volume element on the p-brane associated to the induced riemannian metric g.

3. Deformed equations for BPS configurations

In our discussion we will consider only two types of geometry:

- Infinite flat branes, filling a submanifold $\mathbb{R}^{p+1} \times \{pt\} \subset \mathbb{R}^{p+1} \times M_{9-p}$. These are analyzed in section 4.
- Branes wrapping cycles in manifolds of irreducible non-trivial holonomy. There is a finite number of such cases that preserve supersymmetry and these can be summarized in a table 1.

Cycles marked in the table are those that solve $(1-\Gamma)\eta = 0$ in the absence of B-field and gauge fields on the branes. In this section we investigate how the story changes in the presence of B and gauge fields. In the cases we examine, there are no changes in the conditions on the cycle itself.[3] The gauge fields on the other hand are found (in dimension four or greater) to obey non-linear generalizations of the usual instanton equations.

3.1 SU(2) holonomy

We start by analyzing manifolds of SU(2) holonomy. There are two covariantly constant spinors of the same chirality, η_+ and η_-, and we will choose them in such a way that

$$\gamma_{\bar{m}}\eta_+ = 0\,, \qquad \gamma_{mn}\eta_+ = \Omega_{mn}\eta_-\,. \tag{3.1}$$

With our conventions, the basis for spinors is given by η_+, $\gamma_{mn}\eta_+$ for spinors of negative chirality, and $\gamma_m\eta_+$ for spinors of positive chirality. Notice that $\gamma_{\bar{m}\bar{n}}\eta_- = -\overline{\Omega}_{\bar{m}\bar{n}}\eta_+$. We have normalized $||\Omega||^2 = 4$.

The case $p = 1$. We first consider the case of a one-brane wrapping a 2-cycle. Using the explicit expression for Γ given in (2.6), the equation $\Gamma\eta = \eta$, where η is an $\mathrm{Sl}(2,\mathbb{R})$ doublet of spinors, becomes

$$\begin{aligned}\frac{\sqrt{|g|}}{\sqrt{|g+M|}}\left(1-\frac{1}{2}\gamma^{\mu\nu}M_{\mu\nu}\right)\Gamma_{(0)}\eta_1 &= i\eta_2\,,\\ \frac{\sqrt{|g|}}{\sqrt{|g+M|}}\left(1+\frac{1}{2}\gamma^{\mu\nu}M_{\mu\nu}\right)\Gamma_{(0)}\eta_2 &= i\eta_1\,.\end{aligned} \tag{3.2}$$

The second equation is actually a consequence of the first, as follows from the identity $\Gamma^2 = 1$. The spinors $\eta_{1,2}$ are covariantly constant, so they must be of the form $\eta_1 = z_+\eta_+ + z_-\eta_-$ and $\eta_2 = w_+\eta_+ + w_-\eta_-$, where $z_\pm$, $w_\pm$ are constants. We can always normalize η_1 as $z_\pm = \mathrm{e}^{\mp i\theta}$. Using the above identities for the γ's, and $\gamma_{m\bar{n}}\eta_+ = iJ_{m\bar{n}}\eta_+$, we find that unbroken supersymmetry requires $|w_+|^2 + |w_-|^2 = 2$. We then introduce three angles χ, $\phi_\pm$, and write $w_+ = \sqrt{2}\cos\chi\mathrm{e}^{i\phi_+}$ and $w_- = \sqrt{2}\sin\chi\mathrm{e}^{i\phi_-}$. One finds the following equations:

$$\begin{aligned}f^*(\Omega) &= \frac{i\mathrm{e}^{i\theta}}{\sqrt{2}}(\sin\chi\mathrm{e}^{i\phi_-} + \cos\chi\mathrm{e}^{-i\phi_+})\frac{\sqrt{|g+M|}}{\sqrt{|g|}}\mathrm{vol}_2\,,\\ f^*(J) + iM &= \frac{\mathrm{e}^{i\theta}}{\sqrt{2}}(\cos\chi\mathrm{e}^{i\phi_+} - \sin\chi\mathrm{e}^{-i\phi_-})\frac{\sqrt{|g+M|}}{\sqrt{|g|}}\mathrm{vol}_2\,.\end{aligned} \tag{3.3}$$

[3] We have not however completely ruled out the interesting possibility (raised in [14]) that the cycles themselves are sometimes deformed.

The above equations can be written as

$$\begin{pmatrix} f^*(J)+iM \\ -if^*(\Omega) \end{pmatrix} = U \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\sqrt{|g+M|}}{\sqrt{|g|}} \mathrm{vol}_2 \,, \tag{3.4}$$

where U is a constant U(2) matrix. Notice that, when $M=0$, we obtain the usual calibration condition for a 2-cycle in a K3 manifold that the real vector $(f^*(J), \mathrm{Re}(f^*(\Omega)), \mathrm{Im}(f^*(\Omega)))$ lies on a sphere of radius vol_2 (see, for example, [15, section V.3]):

$$\Big(f^*(J), \mathrm{Re}\big(f^*(\Omega)\big), \mathrm{Im}\big(f^*(\Omega)\big)\Big) = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)\mathrm{vol}_2 \,, \tag{3.5}$$

where θ, ϕ are constant angles along the two-cycle. The $\mathbf{S}^2$ that shows up in (3.5) is in fact related to the $\mathbf{S}^2$ of complex structures. If one chooses a complex structure by choosing a direction, the corresponding point on the sphere gives the holomorphic condition for the 2-cycle, while the intersection of the normal plane to this direction with the sphere gives the $\mathbf{S}^1$ family of special lagrangean submanifolds. Of course, both are related by an SO(3) rotation.

Since M is an antisymmetric tensor in two dimensions, one has $|g+M| = |g| + M^2$. Using this, it is easy to check from (3.4) that:

$$M = \pm \frac{\mathrm{Im}\, u_{11}}{\sqrt{1-(\mathrm{Im}\, u_{11})^2}} \mathrm{vol}_2 \,,$$

$$f^*(J) = \pm \frac{\mathrm{Re}\, u_{11}}{\sqrt{1-(\mathrm{Im}\, u_{11})^2}} \mathrm{vol}_2 \,, \qquad f^*(\Omega) = \frac{i u_{21}}{\sqrt{1-(\mathrm{Im}\, u_{11})^2}} \mathrm{vol}_2 \,, \tag{3.6}$$

where u_{11} and u_{21} are the corresponding entries of the matrix U, and are constant complex numbers. The equations (3.6) say that the vector ($f^*(J)$, $\mathrm{Re}(f^*(\Omega))$, $\mathrm{Im}(f^*(\Omega))$) still has the structure (3.5), even for a nonzero M. Therefore, the Born-Infeld field M does not change the usual calibration condition.

The case $p=3$. For a D3-brane wrapping a four-cycle, the condition $\Gamma\eta = \eta$ of unbroken supersymmetry reduces to

$$\begin{aligned} \frac{\sqrt{|g|}}{\sqrt{|g+M|}} \left(1 + \frac{1}{8}\gamma^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma} + \frac{1}{2}\gamma^{\mu\nu} M_{\mu\nu}\right) \Gamma_{(0)} \eta_1 &= -i\eta_2 \,, \\ \frac{\sqrt{|g|}}{\sqrt{|g+M|}} \left(1 + \frac{1}{8}\gamma^{\mu\nu\rho\sigma} M_{\mu\nu} M_{\rho\sigma} - \frac{1}{2}\gamma^{\mu\nu} M_{\mu\nu}\right) \Gamma_{(0)} \eta_2 &= i\eta_1 \,, \end{aligned} \tag{3.7}$$

where again $\eta = (\eta_1\,, \eta_2)^T$. When D3 wraps the manifold itself, when solving $(1-\Gamma)\eta = 0$ we can take into account that, according to our conventions, $\Gamma_{(0)}\eta = -\eta$. Using the ansatz above for the spinors $\eta_{1,2}$, one finds again that $|w_+|^2 + |w_-|^2 = 2$, and the following equations:

$$\begin{pmatrix} \frac{1}{2}(J+iM)^2 \\ M \wedge \Omega \end{pmatrix} = U \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{\sqrt{|g+M|}}{\sqrt{|g|}} \mathrm{vol}_4 \,, \tag{3.8}$$

where U is again a U(2) matrix.

We have a family of solutions depending on the value of U. For example, if $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then one can check that $M = \mathrm{Re}(\Omega)$ is a solution of the equations. To see this, one has to use that $M^{1,1} = 0$, and that $M \wedge M = 2\,\mathrm{vol}_4$. If U is the identity matrix, one has $M \wedge \Omega = 0$, and $(J + iM)^2/2\sqrt{|g + M|}$ is a constant phase times $d^4\xi$. The first equation says that $M^{2,0} = 0$. Using the fact that $J^2/2$ is the volume element, one sees that $(J + iM)^2/2\sqrt{|g + M|}d^4\xi$ is in fact a complex number of modulus one. For $U = \mathbf{1}$, the above equations can then be written as:

$$\begin{aligned} J \wedge M &= k\left(\mathrm{vol}_4 - \frac{1}{2} M \wedge M\right), \\ M^{2,0} &= 0\,, \end{aligned} \tag{3.9}$$

where k is a constant.

For a compact 4-cycle Σ^4 the value of k in (3.9) is determined in terms of the topological data. Let the closed two forms C_I, $I = 1, 2, \ldots, b_2$ be an integer basis for $H^2(\Sigma^4, R)$, and $I_{IJ} = \int C_I \wedge C_J$ the corresponding intersection matrix. Then in cohomology we can expand

$$\begin{aligned} F &= F^I C_I\,, \qquad & B &= B^I C_I\,, \\ M &= M^I C_I\,, \qquad & J &= J^I C_I\,. \end{aligned} \tag{3.10}$$

We note that $2\pi F^I$ are integrally quantized, but B^I and J^I are not quantized. From the definition of M, $M^I = 2\pi\alpha'(F^I + B^I)$. Integrating (3.9) then yields

$$(I_{KL} J^K J^L - I_{KL} M^K M^L)k = 2 I_{IJ} M^I J^J\,. \tag{3.11}$$

If M and J are orthogonal then either they must have the same norm and k is undetermined, or k must be zero.

Another way of phrasing the conditions (3.9) is that $J + kM$ is a closed $(1,1)$-form such that

$$(J + kM) \wedge (J + kM) = (1 + k^2) J \wedge J\,. \tag{3.12}$$

If we can write $J + kM = T + i\partial\bar{\partial}\phi$, with ϕ a globally well-defined real scalar and T a positive $(1,1)$ form in the cohomology class $[J] + k[M]$ then (3.12) is just the Monge-Ampere equation for ϕ, and there is a unique solution [16].

Yet another form of the conditions (3.9) for supersymmetry can be obtained by decomposing $M = M^- + M^+$ into selfdual and antiselfdual parts. The second equation in (3.9) then implies $M^+ = \phi J$ for some scalar ϕ. The first equation can be solved for ϕ as a function of M^- and k (assuming $k \neq 0$):

$$\phi = -\frac{1}{k}\left(1 \pm \sqrt{1 + k^2(1 + ||M^-||^2)}\right), \tag{3.13}$$

where $\frac{1}{2}M^-\wedge M^- = -||M^-||^2\mathrm{vol}_4$. The three components of M^- are then constrained by the condition $dM = 0$, which becomes

$$dM^- = \mp\frac{k}{2}J\wedge d||M^-||^2\frac{1}{\sqrt{1+k^2(1+||M^-||^2)}}\,. \tag{3.14}$$

3.2 SU(3) holonomy

To analyze the conditions for unbroken supersymmetry, we follow the conventions for covariantly constant spinors of [1]. There are two covariantly constant spinors $\eta_\pm$ of opposite chirality, and conjugate to each other: $\eta_-^* = \eta_+$. They are chosen in such a way that:

$$\gamma_{\bar{m}}\eta_+ = \gamma_m\eta_- = 0\,, \qquad \gamma_{mnp}\eta_+ = \Omega_{mnp}\eta_-\,, \tag{3.15}$$

where m is a holomorphic coordinate index, $m = 1, 2, 3$. The spinor space is spanned by η_+, η_-, $\gamma_m\eta_+$ and $\gamma_{\bar{m}}\eta_-$. We will also need the following identities:

$$\begin{aligned}
\gamma_{\bar{q}mnp}\eta_+ &= \frac{1}{2}\Omega_{mnp}\gamma_{\bar{q}}\eta_-,\\
\gamma_{\bar{q}\bar{m}np}\eta_+ &= (g_{n\bar{m}}g_{p\bar{q}} - g_{n\bar{q}}g_{p\bar{m}})\eta_+,\\
\gamma_{mn}\eta_+ &= \frac{1}{2}\Omega_{mnp}g^{p\bar{q}}\gamma_{\bar{q}}\eta_-\,.
\end{aligned} \tag{3.16}$$

We now analyze the conditions for unbroken supersymmetry for the different cycles of dimension $p+1$.

The case $p = 1$. For a one-brane wrapping a two-cycle we can use again (3.2). Setting, as in the previous subsection, $\eta_1 = z_+\eta_+ + z_-\eta_-$ and $\eta_2 = w_+\eta_+ + w_-\eta_-$, one finds that $(z_+/w_+) = -e^{i\theta}$ is a phase, and the equations read

$$f^*(J) + iM = e^{i\theta}\frac{\sqrt{|g+M|}}{\sqrt{|g|}}\,\mathrm{vol}_2\,, \tag{3.17a}$$

$$dX^m\wedge dX^n\Omega_{mnp} = 0\,, \tag{3.17b}$$

where d denotes the exterior derivative on the worldvolume. In the above equation, and in similar equations in this section, the X^m denote the coordinates of the embedding.

Equation (3.17b) implies that the cycle is holomorphically embedded. A quick way to see this is to use local complex coordinates in the static gauge, and normalize $\Omega_{123} = 1$. If we denote by X^m the complex coordinates for the threefold, the embedding will be described by two functions $X^2 = X^2(X^1, X^{\bar{1}})$, $X^3 = X^3(X^1, X^{\bar{1}})$, where we have identified X^1, $X^{\bar{1}}$ with the complex coordinates on the one-brane worldvolume. Equation (3.17b) says that $\partial_{\bar{1}}X^2 = \partial_{\bar{1}}X^3 = 0$, and the embedding is holomorphic. Therefore, the two-cycles in threefolds are still holomorphic. This implies that $f^*(J) = \mathrm{vol}_2$, and the other equation for BPS configurations (equation (3.17a)) says that M is a constant multiple of the volume form:

$$M = 2\pi\alpha'(F + f^*(B)) = \tan\theta\, \mathrm{vol}_2\,. \tag{3.18}$$

If we fix the topology of the Chan-Paton line bundle: $\int_{\Sigma_2} F = 2\pi n$, and the background field B, the constant $\tan\theta$ is completely determined by integrating the equation (3.18):

$$2\pi n + \int_{\Sigma_2} f^*(B) = \frac{\tan\theta}{2\pi\alpha'}\int_{\Sigma_2} J\,. \tag{3.19}$$

Equation (3.18) represents the only deformation of the usual equations in the presence of B, for $p = 1$ in a threefold. The content is simply that $F/2\pi$ is any integral harmonic form on any holomorphic cycle.

The case $p = 2$. Let's now consider the case of $p = 2$ (i.e. a D2 brane wrapping a 3-cycle in IIA theory). Using the explicit expression (2.6), we find that the deformed supersymmetry equation is:

$$-\frac{i}{\sqrt{|g+M|}}\frac{1}{3!}\epsilon^{\mu\nu\rho}\Big(\gamma_{\mu\nu\rho} + 3M_{\mu\nu}\gamma_\rho\Gamma_{11}\Big)\eta = \eta\,, \tag{3.20}$$

where $\eta = z_+\eta_+ + z_-\eta_-$. We find again that $(z_-/z_+) = -ie^{i\theta}$ is a phase, and the equations read:

$$f^*(\Omega) = e^{i\theta}\frac{\sqrt{|g+M|}}{\sqrt{|g|}}\mathrm{vol}_3\,, \tag{3.21a}$$

$$f^*(J) + iM = 0\,. \tag{3.21b}$$

Since $f^*(J)$ is a real differential form, it follows from the second equation that $M = 0$, and one recovers the special lagrangean condition of [1]. Hence the possibility of gauge field strengths does not lead to new BPS configurations.

The case $p = 3$. For D3 on a four-cycle, using (3.7), we find that unbroken supersymmetry requires $(z_+/w_+)^* = (z_-/w_-) = -ie^{i\theta}$, where θ is a constant and

$$\frac{1}{2}(f^*(J) + iM)\wedge(f^*(J) + iM) = e^{i\theta}\frac{\sqrt{|g+M|}}{\sqrt{|g|}}\,\mathrm{vol}_4\,, \tag{3.22a}$$

$$f^*(\Omega)\wedge dX^{\bar q}g_{\bar q r} + M\wedge dX^m\wedge dX^n\Omega_{mnr} = 0\,. \tag{3.22b}$$

We can see that when $M = 0$, one has $\theta = 0$ as well (just by reality of (3.22b)) and we quickly recover the original condition of [1] that the four-cycle is holomorphic.

In fact even when M is nonzero the four-cycle is holomorphic. As before, we can do the analysis in local complex coordinates. We will assume that the embedding can be described $X^3 = X^3(X^1, X^{\bar{1}}, X^2, X^{\bar{2}})$. At any given point we can always choose a frame in which the metric has the standard form $g = (1/2)\sum_{i=1}^3 (dX^i \otimes dX^{\bar{\imath}} + dX^{\bar{\imath}} \otimes dX^i)$, and $\Omega = dX^1 \wedge dX^2 \wedge dX^3$. As our equations only involve first derivatives, we can work pointwise. In this coordinate system, the second equation of (3.22b) can be written as follows:

$$
\begin{aligned}
\alpha_{\bar{1}} M_{1\bar{2}} - \alpha_{\bar{2}} M_{1\bar{1}} - \alpha_1 M_{\bar{1}\bar{2}} &= \frac{1}{2}\alpha_{\bar{2}}\,, \\
\alpha_{\bar{2}} M_{2\bar{1}} - \alpha_{\bar{1}} M_{2\bar{2}} - \alpha_2 M_{\bar{2}\bar{1}} &= \frac{1}{2}\alpha_{\bar{1}}\,, \\
M_{\bar{1}\bar{2}} &= \frac{1}{2}(\overline{\alpha_{\bar{1}}}\alpha_{\bar{2}} - \overline{\alpha}_2 \alpha_{\bar{1}})\,,
\end{aligned}
\tag{3.23}
$$

where $\alpha_i = \partial_i X^3$, $\alpha_{\bar{\imath}} = \partial_{\bar{\imath}} X^3$, $i = 1, 2$. There is, however, an extra constraint that one has to fulfill: the 2-form M is a real form, and in particular it satisfies $M^*_{1\bar{1}} = -M_{1\bar{1}}$, $M^*_{2\bar{2}} = -M_{2\bar{2}}$. If we write these reality conditions using the explicit expressions in (3.23), we find that the following equation has to be satisfied:

$$
|\alpha_{\bar{1}}|^2 + |\alpha_{\bar{2}}|^2 + |\alpha_1|^2|\alpha_{\bar{2}}|^2 + |\alpha_2|^2|\alpha_{\bar{1}}|^2 - 2\mathrm{Re}(\alpha_1\alpha_{\bar{1}}\overline{\alpha}_2\overline{\alpha}_{\bar{2}}) = 0\,. \tag{3.24}
$$

The sum of the last three terms is greater than or equal to $(|\alpha_1||\alpha_{\bar{2}}| - |\alpha_2||\alpha_{\bar{1}}|)^2$, therefore positive, and it follows then that $\alpha_{\bar{1}} = \alpha_{\bar{2}} = 0$. The embedding is holomorphic, and the four-cycle has to be a divisor.

We can now look at the equations for the gauge field M. From the last equation in (3.23) it follows that $M^{2,0} = 0$. If the cycle is holomorphically embedded, (3.22b) gives an equation for the $(1,1)$ part of M:

$$
f^*(J) \wedge M = \tan\theta \left(\mathrm{vol}_4 - \frac{1}{2} M \wedge M\right). \tag{3.25}
$$

These equations are non-linear deformations of the usual instanton equations $M^{2,0} = 0$, $f^*(J) \wedge M = k\mathrm{vol}_4$, where $f^*(J)$ is the Kähler form on the four-cycle. The BPS configuration we have found is then a divisor in a Calabi-Yau threefold together with a deformed instanton on it. Notice that the above equations are precisely the equations (3.9) that one finds for a D3 brane wrapping a manifold of SU(2) holonomy. Thus, by the discussion surrounding (3.12) there is a unique solution, as long as the cohomology class $[f^*(J) + kM]$ is in the Kähler cone of the 4-cycle.

The case $p = 5$. If the fivebrane wraps the six-cycle itself, we have

$$
\begin{aligned}
\frac{\sqrt{|g|}}{\sqrt{|g+M|}}\left(1 + \frac{1}{8}\gamma^{\mu\nu\rho\sigma}M_{\mu\nu}M_{\rho\sigma} + \frac{1}{2}\gamma^{\mu\nu}M_{\mu\nu} + \frac{1}{48}\gamma^{\mu\nu\rho\sigma\tau\upsilon}M_{\mu\nu}M_{\rho\sigma}M_{\tau\upsilon}\right)\Gamma_{(0)}\eta_1 &= i\eta_2\,, \\
\frac{\sqrt{|g|}}{\sqrt{|g+M|}}\left(1 + \frac{1}{8}\gamma^{\mu\nu\rho\sigma}M_{\mu\nu}M_{\rho\sigma} - \frac{1}{2}\gamma^{\mu\nu}M_{\mu\nu} - \frac{1}{48}\gamma^{\mu\nu\rho\sigma\tau\upsilon}M_{\mu\nu}M_{\rho\sigma}M_{\tau\upsilon}\right)\Gamma_{(0)}\eta_2 &= i\eta_1\,.
\end{aligned}
\tag{3.26}
$$

These equations imply that:

$$\frac{1}{3!}(J+iM)^3 = e^{i\theta}\frac{\sqrt{|g+M|}}{\sqrt{|g|}}\,\mathrm{vol}_6, \tag{3.27a}$$

$$(g^{\bar{q}p} - M^{\bar{q}p})\Omega_{pmn}M^{mn} = 0\,. \tag{3.27b}$$

The last equation can be analyzed as follows. Define $v_p = \Omega_{pmn}M^{mn}$, and consider $v_q^*(g^{\bar{q}p} - M^{\bar{q}p})v_p$, which is zero by (3.27b). Using that M is real and antisymmetric, one finds that $v_q^* g^{\bar{q}p} v_p = 0$, hence $\Omega_{pmn}M^{mn} = 0$. But this means that $M^{2,0} = 0$. We can then write the equations (3.27a) and (3.27b) as

$$\frac{1}{2!}J\wedge J\wedge M - \frac{1}{3!}M\wedge M\wedge M = \tan\theta\left(\mathrm{vol}_6 - \frac{1}{2!}J\wedge M\wedge M\right),$$
$$M^{2,0} = 0\,. \tag{3.28}$$

As in (3.11) the value of the constant θ can be determined in terms of the topological data by integration over the six-cycle.

3.3 G_2 holonomy

To analyze the supersymmetry conditions in manifolds of G_2 holonomy, we need some facts about spinors in such manifolds. We will identify the spinors with the octonions $\mathbb{O}$ in the Cayley-Dickson description: an octonion will be given by a pair of quaternions (a,b), where $a = x^8 + x^1 i + x^2 j + x^3 k$, and $b = x^4 + x^5 i + x^6 j + x^7 k$. The multiplication rule is $(a,b)\cdot(c,d) = (ac - \overline{d}b,\, da + b\overline{c})$, where the overline denotes the usual quaternion conjugation. An octonion written in this way is imaginary if a is an imaginary quaternion (i.e. if $x^8 = 0$ in the above description). The seven imaginary units are then $(i,0),\ldots,(0,k)$. We can identify $\mathbb{R}^7 \simeq \mathrm{Im}\,\mathbb{O}$ in the obvious way, and Clifford multiplication is therefore given by octonionic multiplication by the imaginary units. The γ matrices will be i times the imaginary units acting through multiplication, in order to have $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ in flat space. They are then 8×8 imaginary, antisymmetric matrices.

In a manifold of G_2 holonomy there is a covariantly constant spinor ϑ which in the above representation can be taken as $(1,0)$ (i.e. the unit octonion). A basis for the spinor space is then given by ϑ, $\gamma_\mu\vartheta$, $\mu = 1,\ldots,7$. We also have a calibration Φ, which is a closed three-form, and the following identities:

$$\gamma_{\mu\nu\rho}\vartheta = i\Phi_{\mu\nu\rho}\vartheta\,, \qquad \gamma_{\mu\nu}\vartheta = \Phi_{\mu\nu\rho}\gamma^\rho\vartheta\,, \qquad \gamma_{\mu\nu\rho\sigma}\vartheta = (*\Phi)_{\mu\nu\rho\sigma}\vartheta\,. \tag{3.29}$$

We can already analyze the conditions for unbroken supersymmetry in the presence of Born-Infeld fields. For a D2 brane, we get the equations:

$$f^*(\Phi) = \frac{\sqrt{|g+M|}}{\sqrt{|g|}}\,\mathrm{vol}_3\,, \tag{3.30a}$$

$$M\wedge dX^\mu = 0\,. \tag{3.30b}$$

The second equation implies $M = 0$. Hence we recover the usual condition for an associative cycle in a G_2 manifold.

The analysis for a D3 brane is similar, and we obtain the following conditions:

$$f^*(*\Phi) - \frac{1}{2} M \wedge M = \frac{\sqrt{|g+M|}}{\sqrt{|g|}} \mathrm{vol}_4 \,, \tag{3.31a}$$

$$M \wedge dX^\mu \wedge dX^\nu \Phi_{\mu\nu\rho} = 0 \,. \tag{3.31b}$$

The equation (3.31b) can be interpreted as follows. The **21** of Spin(7) decomposes under G_2 as $\mathbf{21} = \mathbf{14} + \mathbf{7}$. (3.31b) says that the 2-form $M \wedge dX^\mu \wedge dX^\nu$ belongs to the **14**. Using the projector of the **21** of Spin(7) onto the **7** [17]

$$\mathbf{P}_{\mu\nu\rho\sigma} = \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - (*\Phi)_{\mu\nu\rho\sigma}) \,, \tag{3.32}$$

we can write (3.31b) as

$$\mathbf{P}(M \wedge dX^\mu \wedge dX^\nu) = 0 \,. \tag{3.33}$$

In the case when the 4-cycle is a coassociative 4-fold the equation (3.33) implies that M is an anti-self-dual 2-form on the D3 brane worldvolume: $M^+ = 0$. This can be easily proved by working in local coordinates. Another proof proceeds as follows. If the cycle is coassociative then we may replace $f^*(*\Phi) = \mathrm{vol}_4$, and divide through by vol_4. We then square the equation and use

$$\frac{|g+M|}{|g|} = 1 - \frac{1}{2}\,\mathrm{Tr}\, M^2 + \det M \tag{3.34}$$

(here M is an antisymmetric matrix, in local coordinates). Then the square of (3.31a) becomes $(1 - \mathrm{Pf}(M))^2 = 1 - \frac{1}{2}\,\mathrm{Tr}(M^2) + \det M$ so $\mathrm{Tr}(M^+)^2 = 0$.

When a D6 brane wraps a G_2 manifold, the conditions for unbroken supersymmetry give equations for the gauge field. These will be analyzed in section 4 using the group-theory approach.

3.4 SU(4) holonomy

On a manifold of SU(4) holonomy there are two covariantly constant spinors, $\eta_\pm$, with the same chirality and complex conjugate to each other. They are chosen in such a way that $\gamma_{\bar{m}}\eta_+ = 0$. We have the following identities,

$$\gamma_{mnpq}\eta_+ = \Omega_{mnpq}\eta_- \,, \qquad \gamma_{\bar{q}mnp}\eta_+ = 3iJ_{\bar{q}[m}\gamma_{np]}\eta_+ \,. \tag{3.35}$$

The positive chirality spinor space is spanned by η_+, $\gamma_{mn}\eta_+$, and η_-, while the negative chirality spinor space is spanned by $\gamma_m\eta_+$, $\gamma_{mnp}\eta_+$. Supersymmetric cycles in fourfolds were recently considered in [18].

The case $p = 1$. For a D1 brane wrapping a 2-cycle in a Calabi-Yau fourfold, we find two equations. The first one is (3.17a), while the second one is

$$dX^m \wedge dX^n = 0\,. \tag{3.36}$$

This implies that the cycle is holomorphic. M is constrained in an analogous way to the SU(3) case.

The case $p = 3$. For a D3 brane wrapping a four-cycle, the conditions we obtain are more complicated. Set $z_\pm = \mathrm{e}^{\mp i\theta/2}$. The analysis of the equations gives $w_+ = i\mathrm{e}^{i\phi/2}$, $w_- = i\mathrm{e}^{-i\phi/2}$, and the following conditions:

$$-\tfrac{1}{2}(f^*(J) + iM)^2 + f^*(\overline{\Omega}_\theta) = \mathrm{e}^{i(\phi+\theta)/2}\frac{\sqrt{|g+M|}}{\sqrt{|g|}}\,\mathrm{vol}_4\,, \tag{3.37a}$$

$$\mathrm{Im}(\mathrm{e}^{i(\phi+\theta)/2} f^*(\Omega_\theta)) = 0\,, \tag{3.37b}$$

$$(f^*(J) + iM) \wedge \left(dX^n \wedge dX^p + \tfrac{1}{2}(\overline{\Omega}_\theta)_{\bar{p}\bar{q}}{}^{np} dX^{\bar{p}} \wedge dX^{\bar{q}} \right) = 0\,. \tag{3.37c}$$

where $\Omega_\theta := \mathrm{e}^{-i\theta}\Omega$. To write the last equation, we have used that $\gamma_{\bar{m}\bar{n}}\eta_- = -\frac{1}{2}\overline{\Omega}_{\bar{m}\bar{n}}{}^{pq}\gamma_{pq}\eta_+$, with the normalization $||\Omega||^2 = 16$. Again, notice that when $M = 0$ the reality of $f^*(J)$ imposes that $\theta + \phi = 0$. The second equation gives $\mathrm{Im}(\Omega_\theta) = 0$ and the first equation reads,

$$-\frac{1}{2} f^*(J)^2 + \mathrm{Re}(f^*(\Omega_\theta)) = \mathrm{vol}_4\,, \tag{3.38}$$

which is the usual condition for a Cayley calibration [15] obtained in this context in [19].

The case $p = 5$. Finally, we can analyze the deformed equations for a six-cycle in a fourfold. If we set $z_\pm = \mathrm{e}^{\mp i\theta/2}$, we find again $w_+ = i\mathrm{e}^{i\phi/2}$, $w_- = i\mathrm{e}^{-i\phi/2}$, and three equations which are similar to (3.37c):

$$\tfrac{1}{3!}(f^*(J) + iM)^3 - iM \wedge f^*(\overline{\Omega}_\theta) = -i\mathrm{e}^{i\frac{(\phi+\theta)}{2}}\frac{\sqrt{|g+M|}}{\sqrt{|g|}}\,\mathrm{vol}_6\,, \tag{3.39a}$$

$$\mathrm{Im}(M \wedge \mathrm{e}^{i(\phi+\theta)/2} f^*(\Omega_\theta)) = 0\,, \tag{3.39b}$$

$$(f^*(J) + iM)^2 \wedge \left(dX^n \wedge dX^p + \tfrac{1}{2}(\overline{\Omega}_\theta)_{\bar{p}\bar{q}}{}^{np} dX^{\bar{p}} \wedge dX^{\bar{q}} \right) = 0\,. \tag{3.39c}$$

where again $\Omega_\theta := \mathrm{e}^{-i\theta}\Omega$. Notice that when $M = 0$ one recovers the usual conditions for a holomorphic embedding (i.e. the cycle is a divisor).

For a D7 brane wrapping a SU(4) manifold, the analysis of the equations along these lines is more involved. As it will become clear in section 4, one finds a natural generalization of (3.28) that can be obtained much more easily using the group-theory approach.

3.5 Spin(7) holonomy

To analyze the unbroken supersymmetries in manifolds of Spin(7) holonomy, we first set the relevant spinor algebra. We will regard the spinors of positive or negative chirality as octonions: $S^+ \simeq S^- \simeq \mathbb{O}$. The Clifford algebra $C\ell_8$ is represented by

$$\begin{aligned}\Sigma^{1,\dots,8} &= \begin{pmatrix} 0 & -i\gamma^{1,\dots,8} \\ -i\tilde{\gamma}^{1,\dots,8} & 0 \end{pmatrix}, \\ \tilde{\gamma}^{1,\dots,7} &= -\gamma^{1,\dots,7}, \\ \tilde{\gamma}^{8} &= \gamma^{8}, \end{aligned} \tag{3.40}$$

where γ^i is the representation of $C\ell_7$ described above, and $\tilde{\gamma}^{1,\dots,7}$ is the other inequivalent representation of $C\ell_7$ (notice that $-i\gamma^{1,\dots,7}$ is given by octonionic multiplication by the imaginary units). We take $\gamma^8_{a\dot{a}} = i\delta_{a\dot{a}}$. The chirality operator is

$$\bar{\Sigma} = \begin{pmatrix} -1_8 & 0 \\ 0 & 1_8 \end{pmatrix} \tag{3.41}$$

Note that $(\Sigma^i)^T = \Sigma^i$, and $(\Sigma^i)^* = \Sigma^i$, $i = 1, \dots, 8$.

We choose the embedding of Spin(7) in Spin(8) of [17], in which the spinor representation decomposes as $\mathbf{8}_s \to \mathbf{1} + \mathbf{7}$.

In a manifold of Spin(7) holonomy there is one covariantly constant spinor, which we will take of positive chirality. We will denote it by ϑ again, and using the identification $S^+ \simeq \mathbb{O}$ this spinor can be regarded as the unit octonion. We also have a calibration Ω, which in this case is a closed four-form. In terms of the calibration on manifolds of G_2 holonomy, we have (in an orthonormal basis):

$$\Omega = \Phi \wedge dx^8 + *\Phi\,. \tag{3.42}$$

Notice that this differs from the conventions in [15]. The calibration satisfies $*\Omega = \Omega$.

The positive chirality spinor space is spanned by ϑ and $\Sigma_{\mu\nu}\vartheta$, $\mu, \nu = 1, \dots, 8$. It is important to notice that the $\Sigma_{\mu\nu}$ are not independent: they obey the self-duality condition

$$\Sigma_{\mu\nu}\vartheta = -\frac{1}{6}\Omega_{\mu\nu}{}^{\rho\sigma}\Sigma_{\rho\sigma}\vartheta\,. \tag{3.43}$$

This means that the tensor $\Sigma_{\mu\nu}\vartheta$ in the $\mathbf{28}$ of Spin(8) belongs to the $\mathbf{7}$ of Spin(7). Therefore, only seven components are linearly independent and we find the right counting of generators for S^+. The generators of S^- are simply given by $\Sigma_\mu\vartheta$, $\mu = 1, \dots, 8$. To analyze the unbroken supersymmetries we will need the following identity

$$\Sigma_{\mu\nu\rho\sigma}\vartheta = \Omega_{\mu\nu\rho\sigma}\vartheta\,, \tag{3.44}$$

and the expression for the projector $\mathbf{28} \to \mathbf{7}$ [17]:

$$\mathbf{P}_{\mu\nu\rho\sigma} = \frac{1}{8}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} - \Omega_{\mu\nu\rho\sigma})\,. \tag{3.45}$$

We can now analyze a D3 brane wrapping a four-cycle in a manifold of Spin(7) holonomy. After some straightforward algebra, one finds

$$f^*(\Omega) - \frac{1}{2}M \wedge M = \frac{\sqrt{|g+M|}}{\sqrt{|g|}}\,\mathrm{vol}_4\,, \tag{3.46a}$$

$$\mathbf{P}(M \wedge dX^\mu \wedge dX^\nu) = 0\,. \tag{3.46b}$$

Notice that, for $M = 0$, we recover the fact a supersymmetric cycle is Cayley [19].

In a manner similar to the case of G_2 holonomy, in the case when the 4-cycle is a Cayley 4-fold the equation (3.46b) implies that M is an anti- self-dual 2-form on the D3 brane worldvolume: $M^+ = 0$. Again, this can be easily proved by working in local coordinates, or using exactly the same argument as in the G_2 case by squaring (3.46a).

Again, the case of the D7 brane wrapping a Spin(7) manifold is more involved using these techniques, and will be considered in section 4.

3.6 A comment on the equations

To write the above conditions for deformed cycles, we have decomposed $\Gamma\eta = \eta$ in the appropriate basis of the spinor space and we have written the equations that one derives for the different independent elements involved in the equations. For example (3.17a) gives the piece proportional to η_+, while (3.17b) give the piece proportional to $\gamma^p\eta_+$. However, as it has been pointed out in [1, 19], to find solutions of this equation it is enough to solve the equation $\eta^\dagger\Gamma\eta = \eta^\dagger\eta$, which usually gives only one equation (as the inner product with $\eta^\dagger$ kills the components which are orthogonal to η). The reason that these two procedures are equivalent is the following. If we denote $P_- = \frac{1}{2}(1-\Gamma)$, we see from the properties of Γ that P_- is an hermitean projector, $P_-^\dagger = P_-$, $P_-^2 = P_-$. The condition for unbroken supersymmetry can be written as $P_-\eta = 0$, but this is equivalent to

$$\big(P_-\eta\big)^\dagger P_-\eta = \eta^\dagger P_-\eta = 0\,. \tag{3.47}$$

Therefore, solving $P_-\eta = 0$ is equivalent to solving $\eta^\dagger P_-\eta = 0$. This implies, in particular, that the additional equations in $P_-\eta = 0$ are in fact consequences of $\eta^\dagger P_-\eta = 0$. This last equation gives the conditions labeled as (a) in this section. For $M = 0$, they give the standard definitions of calibrations. The fact that the other equations (labeled as (b) and (c)) follow from this one is not obvious from a mathematical point of view. In the $M = 0$ case, they give additional properties of the calibrated submanifolds. For example, for $p = 1$ (3.17b) is equivalent to holomorphicity, which in turn is implied by (3.17a) when $M = 0$. On the other hand, these additional equations can show features which are not manifest in the main equations (a). This is one of the reasons that we have decided to spell them out in detail. As we will see in the next section, they can be extremely useful once the M field is included. The mathematical meaning of equations (b), (c), at least for the standard calibrations, is the following: the condition that a submanifold is calibrated can be stated in terms of a differential system ψ_j, $j = 1, \ldots, n$, where the ψ_j are differential forms on the ambient space [15]. A submanifold W is calibrated if and only if the forms ψ_j restrict to zero on W. The equations (b), (c) that we have found are in fact part of the system of equations associated to this differential system.

As a final remark, notice the appearance in the deformed equations of the complexified Kähler form, since $f^*(J) + iM = f^*(J + 2\pi i\alpha' B) + 2\pi i\alpha' F$.

3.7 Instanton actions

One application of this work is to further study of mirror symmetry. In particular, in [1] mirror symmetric formulae for sums over D-brane and M-brane instantons were suggested. It remains a challenging problem to make these formulae concrete and test them. One important ingredient in the sums over D-brane instanton corrections are the instanton actions. It is worth noting that the real part of the instanton actions can easily be derived from the above equations.[4] As an illustration consider the D-instanton effects in IIB string theory on a CY 3-fold. We must consider $p = -1, 1, 3, 5$. The case $p = -1$ has not been discussed since it doesn't lead to interesting worldvolume equations. For the case $p = 1$ wrapping a 2-cycle W_2 we integrate equation (3.17a) and then substitute the result (3.19) to get:

$$\mathrm{Re}(I) = -T_1 \sqrt{\left(\int_{W_2} J\right)^2 + \left(\int_{W_2} M\right)^2} \tag{3.48}$$

as expected from the tension formula for the $\mathrm{SL}(2,\mathbb{Z})$ multiplet of strings. Similarly, for a D3 wrapping a 4-cycle W_4 we get, in a similar way

$$\mathrm{Re}(I) = -T_3 \sqrt{\left(\int_{W_4} J \wedge M\right)^2 + \left(\int_{W_4} \frac{1}{2} J \wedge J - \frac{1}{2} M \wedge M\right)^2} \tag{3.49}$$

in accord with the 2-brane and 0-brane charges induced by the Chan-Paton bundle. Finally, for a D5 wrapping the full Calabi-Yau 3-fold

$$\mathrm{Re}(I) = -T_5 \sqrt{\left(\int_{W_6} \frac{1}{3!} J^3 - \frac{1}{2!} J M^2\right)^2 + \left(\int_{W_6} \frac{1}{2} J^2 M - \frac{1}{3!} M^3\right)^2}\,. \tag{3.50}$$

Again, this is in accord with the standard formulae for induced D-brane charges from the Chan-Paton bundle, to leading order in α'.

[4] The imaginary part, by contrast, is a much more subtle quantity, and is discussed in [20, 21].

JHEP01(2000)005

4. Group-theoretical basis for the deformed instanton equations

Since we have found that the cycles are not deformed, we now consider euclidean flat branes wrapping a submanifold $\mathbb{R}^{p+1} \times \{pt\} \subset \mathbb{R}^{p+1} \times M_{9-p}$ and the deformation of the instanton equation on them. We will exploit here the fact that the Γ matrix can be written in the rotated form (2.9). We want to solve the equation (2.4), where η, in the type-IIB theory, is a doublet of spinors η_i, $i = 1, 2$, and Γ depends on the Born-Infeld field M. Suppose that we find a covariantly constant spinor χ in an irreducible representation of $\mathrm{Spin}(p+1)$ satisfying the equation

$$\frac{1}{2} Y_{ij} \gamma^{ij} \chi = k \chi \,, \tag{4.1}$$

where k is a constant scalar. Then, the equation (2.4) is easily solved by setting $\eta_1 = \chi$, $\eta_2 = \pm i^{(p+3)/2} \mathrm{e}^k \chi$ (where the sign depends on the chirality of χ).

The equation (4.1) is a simple equation for unbroken supersymmetry in terms of the field Y. However, since there is a nonlinear relation between Y and M given by (2.14), the conditions for unbroken supersymmetry of M can be complicated. M on the other hand obeys the simple relation $dM = 0$, which is a complicated constraint on Y.

Some understanding of the relation (4.1) follows simply from group theory. Let us regard the antisymmetric matrix Y_{ij} (in a local orthonormal frame) as an element of the Lie algebra $\mathrm{spin}(p+1)$. The equation (4.1) simply says that the infinitesimal rotation by Y preserves χ up to a rescaling. Let $h_{\|} \subset \mathrm{Spin}(n)$ be the Lie subalgebra stabilizing the one-dimensional space spanned by χ. Thus elements of $h_{\|}$ act on χ by a (possibly vanishing) constant. Let $h_{\perp}$ be the orthogonal complement of h in the Killing metric. Elements in $h_{\perp}$ rotate χ to a nonzero orthogonal spinor. Then

$$\mathrm{spin}(n) = h_{\|} \oplus h_{\perp} \,. \tag{4.2}$$

The equations (4.1) simply say that $Y_{\perp} = 0$, and $Y_{\|} = \text{constant}$, in an obvious notation. Now these translate into conditions on $(\tanh Y)_{\|}$ and $(\tanh Y)_{\perp}$. The relation between $(\tanh Y)_{\|}$ and $(\tanh Y)_{\perp}$ is complicated in general, although it is constrained by group theory.

In the following subsections we will analyze the condition (4.1) and the resulting equations for M, in various dimensions, using this group-theoretic approach.

4.1 $p = 3$ case and deformed instanton equation in four dimensions

We begin with the case of $p = 3$ in four-dimensional euclidean space with metric $g_{\mu\nu} = \epsilon \delta_{\mu\nu}$. The Lie algebra $\mathrm{so}(4)$ is the representation $(3,1) \oplus (1,3)$ of $\mathrm{SU}(2) \oplus \mathrm{SU}(2)$. The choice of spinor singles out the stabilizer subgroup $h = (\mathrm{U}(1), 0)$. Let Y be a 4×4 antisymmetric real matrix. Define $Z := \tanh Y$ by the power series $Z =$

$\sum_{m=0}^{\infty} a_m Y^{2m+1}$ (a_m can be written in terms of Bernoulli numbers, but we will not need this). Note that Z is antisymmetric and real. For an antisymmetric matrix, we denote by $Y = Y^+ + Y^-$ the separation into selfdual and antiselfdual pieces. This is the projection to $(3,1)$, and $(1,3)$, respectively. We claim that:

$$\frac{(\tanh Y)^+}{1 - \mathrm{Pf}(\tanh Y)} = \frac{1}{2}\tanh(2Y^+)\,. \tag{4.3}$$

Proof. We first note that (4.3) is SO(4) invariant. For Y real antisymmetric there is always an SO(4) rotation that skew diagonalizes it, so $Y = y_{12}T_{12} + y_{34}T_{34}$, where $T_{ij} := e_{ij} - e_{ji}$, and e_{ij} are matrix units. Then

$$Y^+ = \frac{1}{2}(y_{12} + y_{34})(T_{12} + T_{34})\,. \tag{4.4}$$

In this skew diagonal form one easily checks (4.3) by direct computation. Note that if Y is skew diagonal then

$$\tanh Y = \tan(y_{12})T_{12} + \tan(y_{34})T_{34}\,. \tag{4.5}$$

Now we use the addition formula for tangents.

Now, the solution to the equations

$$Y^{ij}\gamma_{ij}\eta = k\eta\,, \tag{4.6}$$

where k is a constant and η is of negative chirality, is that $Y = Y^+ + Y^-$ where Y^+ is a constant in the stabilizer subgroup $h = \mathrm{U}(1)$. (This constant can be expressed in terms of k) while the component Y^- is an arbitrary function of spacetime. Thus, it follows from (4.3) and $M = \epsilon\,\tanh(Y)$[5] that

$$\frac{\xi^{-1}(F+B)^+}{1 - \xi^{-2}\,\mathrm{Pf}(F+B)} = \frac{1}{2}\tanh(2Y^+) = \text{const.}\,, \tag{4.7}$$

where we have introduced the parameter

$$\xi = \frac{\epsilon}{2\pi\alpha'}\,. \tag{4.8}$$

The constant in (4.7) is evaluated by going to infinity (recall that we are now in a noncompact situation). As in [6], we will consider configurations in which $F \to 0$ at infinity and B is constant, corresponding to localized instantons. This fixes the gauge freedom relating F and B completely, and we obtain:

$$\frac{(F+B)^+}{\mathrm{Pf}(F+B) - \xi^2} = \frac{B^+}{\mathrm{Pf}(B) - \xi^2}\,. \tag{4.9}$$

We can now compare this equation with what we found in section 3. In order to do this, we have to specify the appropriate complex structure. In the complex structure induced by the reduced holonomy in (3.9) or (3.25), the B field is a $(1,1)$ form. Therefore, by choosing a complex structure for (4.9) in which B is also of type $(1,1)$, we get again (3.8).

[5]The factor of ϵ arises because we have kept $M_{\mu\nu}$ fixed while scaling the flat metric by ϵ. This rescales $M_{\mu\nu}$ by ϵ.

4.2 Deformation of the hermitean Yang-Mills equations: $p = 5$ and $p = 7$

We will now analyze the equation (4.1) on $\mathbb{R}^6$. This equation says that the infinitesimal rotation by Y preserves ϵ up to a constant. The covariantly constant spinor breaks the local frame group $\mathrm{SO}(6) \to \mathrm{SU}(3)$ at every point, and chooses a complex structure. Relative to this complex structure we have the deformed equations:

$$\begin{aligned} Y^{2,0} &= 0\,, \\ J^{m\bar{n}} Y_{m\bar{n}} &= k\,, \end{aligned} \tag{4.10}$$

where k is a constant.

Now we can analyze the meaning of the equations for $M/\epsilon = \tanh Y$ along the lines of the previous subsection. Y is now a 6×6 antisymmetric matrix. Without loss of generality we can take the local frame components of J to be given by $J = T_{12} + T_{34} + T_{56}$, so that $\mathrm{U}(3) \subset \mathrm{SO}(6)$ is defined by $\{A : AJ = JA, A \in \mathrm{SO}(6)\}$. Under this embedding the antisymmetric rep **15** of SO(6) decomposes as:

$$\mathbf{15} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}^* \oplus \mathbf{8}\,. \tag{4.11}$$

The matrix Y decomposes then as:

$$Y = Y_1 + Y_3 + Y_{3^*} + Y_8\,. \tag{4.12}$$

The equations (4.10) say that $Y_3 = Y_{3^*} = 0$ and Y_1 is a constant times the identity, while Y_8 is an arbitrary undetermined function.

The matrix $Z := \tanh Y$ is also real antisymmetric and hence decomposes as $Z = Z_1 + Z_3 + Z_{3^*} + Z_8$. The Z_i's are nonlinear functions of the Y_i's. However, if $Y_3 = 0$ then $Z_3 = 0$. This follows because in the power series representation of Z, if Y has zero triality then Z has zero triality. The conclusion of this analysis is that $Y^{2,0} = 0$ implies $M^{2,0} = 0$. Thus, "most" of the 6d hermitean Yang-Mills equations are undeformed.

Moreover, if $Y_3 = Y_{3^*} = 0$ then we have the identity:

$$\frac{(\tanh Y)_1 - \frac{1}{3}\,\mathrm{Pf}(\tanh Y) J}{1 - \frac{1}{2}\,\mathrm{Pf}(\tanh Y)\mathrm{Tr}(J(\tanh Y)^{-1})} = \frac{1}{3}\tanh(3Y_1)\,. \tag{4.13}$$

The left hand side defines a nonlinear function of $M/\epsilon = \tanh Y$. Evaluating the constant by going to infinity we get a deformation of the 6d hermitean Yang-Mills equation, which can be written as follows:

$$\frac{\xi^2(\mathcal{F} \wedge J^2)/2! - \mathcal{F}^3/3!}{\xi^2 J^3/3! - (\mathcal{F}^2 \wedge J)/2!} = \frac{\xi^2(B \wedge J^2)/2! - B^3/3!}{\xi^2 J^3/3! - (B^2 \wedge J)/2!}\,, \tag{4.14}$$

In this equation, $\mathcal{F} = F + B$, we have assumed that B is of type $(1,1)$, and we have used the fact that $\mathcal{F}^{2,0} = 0$. Notice that all the terms in (4.14) are proportional to the volume form, $J^3/3!$, and the quotient of forms should be understood as a quotient of the corresponding scalars. (4.14) can can also be written as

$$\frac{\left[e^J \sin(\xi^{-1}\mathcal{F})\right]^{\text{top}}}{\left[e^J \cos(\xi^{-1}\mathcal{F})\right]^{\text{top}}} = \frac{\left[e^J \sin(\xi^{-1}B)\right]^{\text{top}}}{\left[e^J \cos(\xi^{-1}B)\right]^{\text{top}}}, \tag{4.15}$$

where the superscript top means that we take the top form in the expansion, which in complex dimension n has degree $2n$. Finally, notice that (4.14) agrees with (3.27a) and (3.27b) (if, as we explained in the four-dimensional case, B is of type $(1,1)$). Also notice that the above deformed equation solves (3.39a)–(3.39c) provided that the embedding is holomorphic: (3.39c) holds if $M^{2,0} = 0$, and, since $f^*(\Omega) = 0$, (3.39a) reduces to (4.14) (for the value of θ fixed by the behavior at infinity).

Equation (4.10) straightforwardly generalizes to the next case $p = 7$ and further analysis gives as before $M^{2,0} = 0$ (by virtue of the decomposition under the embedding of U(4) into SO(8): $\mathbf{28} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{6}^* \oplus \mathbf{15}$ and four-ality). Moreover, we get again (4.15), where the top form has degree 8. Explicitly, we find:

$$\frac{\xi^3(\mathcal{F} \wedge J^3)/3! - \xi(\mathcal{F}^3 \wedge J)/3!}{\xi^4 J^4/4! - \xi^2(\mathcal{F}^2 \wedge J^2)/(2!)^2 + \mathcal{F}^4/4!} = \frac{\xi^3(B \wedge J^3)/3! - \xi(B \wedge J)/3!}{\xi^4 J^4/4! - \xi^2(B \wedge J^2)/(2!)^2 + B^4/4!}. \tag{4.16}$$

In a manifold of SU(4) holonomy, the equation for the deformed instanton are also $M^{2,0} = 0$ and (4.16), with the only difference that the constant in the right hand side is evaluated in terms of the topological data, as in (3.11).

4.3 G_2 and Spin(7)

As a further application let us consider the possible deformation of the G_2-instanton equations. Here Y is in the **21** of Spin(7) and a nonzero constant spinor χ determines an embedding of G_2 into Spin(7). For example, from χ we get $\gamma_{ijk}\chi = i\Phi_{ijk}\chi$ as in (3.44), and the G_2 subgroup is the subgroup of Spin(7) preserving Φ_{ijk}. Now $Y = Y_7 + Y_{14}$. The equations state that $Y_7 = 0$. This is equivalent to $(\tanh Y)_7 = 0$ because tensor products of the **14** never produce the **7**. (One way to see this is to note that weights in the tensor products of the **14** are always integer sums of root vectors of G_2, so one can never produce the weights of the **7** this way.)

In exactly the same way we find that the Spin(7) equations are undeformed. That is, the component of M in the **7** of Spin(7) is zero.

5. Relations to noncommutative instanton equations

In this section, we will analyze the deformed instanton equations of the previous section, focusing on their limiting behavior. In this way we will recover and generalize some of the results in [6].

In the deformed equations of the previous section there are two parameters, α' and ϵ, which are combined into the parameter ξ defined in (4.8). We now consider several different limiting behaviors. One important limit is to take $\alpha' \to 0$ while keeping ϵ fixed (and therefore $\xi \to \infty$). This is the usual zero slope limit. Another interesting limit is discussed in [6, 22]. In [6], Seiberg and Witten have shown that the relation of string theory to noncommutative geometry appears in the double scaling limit $\epsilon \to 0$, $\alpha' \sim \epsilon^{1/2} \to 0$. In this limit (that will be called the Seiberg-Witten limit), $\xi \sim \epsilon^{1/2} \to 0$. Finally, we can consider the limit in which the B field is set to zero. We will now analyze these limits in the different situations that we have considered.

5.1 Instanton equation in four dimensions

We will first analyze the deformed instanton equation (4.9). In the zero slope limit, $\xi \to \infty$, and we get the usual instanton equation $F^+ = 0$. For $B = 0$, the equation reduces again to the instanton equation, even for finite ξ. This is in agreement with the observations in [6, section 2.3]. In order to make contact with the deformed instanton equations that correspond to noncommutative instantons, we have to take the Seiberg-Witten limit $\xi \to 0$. We find,

$$\frac{(F+B)^+}{\mathrm{Pf}(F+B)} = \frac{B^+}{\mathrm{Pf}(B)}\,, \tag{5.1}$$

which is precisely [6, equation (4.45)]. In fact, our equation (4.9) can be regarded as a one-parameter deformation of the Seiberg-Witten equation, where the deformation parameter is ξ. It is also illuminating to write (4.9) in the open string frame, following the discussion in [6]. The open string metric in the zero-slope limit is given by:

$$G_{ij} = -\frac{(2\pi\alpha')^2}{\epsilon}(B^2)_{ij}\,, \tag{5.2}$$

which can be obtained from the vierbein

$$E = -\frac{\epsilon^{1/2}}{2\pi\alpha'}B^{-1} \tag{5.3}$$

as $G = (EE^t)^{-1}$. In the open string frame, one has:

$$\begin{aligned} F^+ &= \frac{B^+}{4\mathrm{Pf}(B)}\epsilon^{ijkl}B_{ij}F_{kl} - (\mathrm{Pf}(E))^{-1}E^tF^+_GE\,, \\ B^+ &= (\mathrm{Pf}(E))^{-1}E^tB^+_GE\,, \end{aligned} \tag{5.4}$$

where F^+_G, B^+_G denote the self-dual projections in the open string metric (5.2). Taking into account that $\theta^{ij} = (B^{-1})^{ij}$, we find the equation:

$$F^+_G = \frac{1}{4}\frac{1}{1-\xi^2\mathrm{Pf}(\theta)}(G\theta^+_GG)\big(\widetilde{F}^{ij}F_{ij} + 2\xi^2\mathrm{Pf}(\theta)\widetilde{F}^{ij}\theta^{-1}_{ij}\big)\,, \tag{5.5}$$

where

$$\widetilde{F}^{ij} = \frac{1}{2}\frac{\epsilon^{ijkl}}{\sqrt{\det G}}F_{kl}\,. \tag{5.6}$$

Eq. (5.5) can be regarded as a two-parameter deformation of the usual instanton equation, where the parameters are now θ and ξ.

One can study spherically symmetric solutions of (5.5) along the lines of [6, equations (4.56) to (4.62)]. This exercise is subject to the criticism (discussed in [6]) that the instanton equations are not valid near $r \to 0$ because the fields are varying too rapidly there. One finds a one parameter family of solutions interpolating between $h \sim 1/r^2$ and $h \sim 1/r^4$. It might be interesting to investigate solutions to the nonabelian generalizations of these equations, because these might be nonsingular.

5.2 Hermitean Yang-Mills equations

We now study an analogous deformation of the hermitean Yang-Mills equations using the equations we have found above for $p = 5, 7$. First of all, in the zero slope limit $\xi \to \infty$ we recover the ordinary hermitean Yang-Mills equations, $F^{2,0} = 0$, $F \wedge J^{\frac{p-1}{2}} = 0$, as expected (we assume that B is of type $(1,1)$). On the other hand, when $B = 0$ and ξ is finite we do not recover these equations. Rather

$$[\mathrm{e}^J \sin(\xi^{-1}F)]^{\mathrm{top}} = 0\,, \tag{5.7}$$

is still a deformation of hermitean Yang-Mills. For example, for $p = 5$ we find

$$\xi^2\frac{J^2}{2!} \wedge F - \frac{1}{3!}F^3 = 0\,. \tag{5.8}$$

If the reasoning of [6] extends to this case, then we should have a one-to-one correspondence between solutions of (5.7) and solutions of the usual hermitean Yang-Mills equations in six dimensions.

In the Seiberg-Witten limit $\xi \to 0$ we get the equations:

$$\frac{\mathcal{F}^{\frac{p-1}{2}} \wedge J}{\mathrm{Pf}(\mathcal{F})} = \frac{B^{\frac{p-1}{2}} \wedge J}{\mathrm{Pf}(B)}\,, \tag{5.9a}$$

$$\mathcal{F}^{2,0} = 0\,. \tag{5.9b}$$

Using that $\phi^{\frac{p+1}{2}} = (\mathrm{Pf}(\phi))J^{\frac{p+1}{2}}$, $\phi^{\frac{p-1}{2}} \wedge J = \mathrm{Pf}(\phi)\,\mathrm{Tr}(J\phi^{-1})$, where ϕ is a $(1,1)$ form, we obtain:

$$\mathrm{Tr}\left[J\left(\frac{1}{F+B} - \frac{1}{B}\right)\right] = 0\,. \tag{5.10}$$

To compare with the non-commutative YM equations we should go to the open string frame and recall that, ignoring the terms involving derivatives in F, one has [6]:

$$\hat{F} = \frac{1}{1+F\theta}F = -B\left(\frac{1}{B+F} - \frac{1}{B}\right)B\,, \tag{5.11}$$

where $\hat{F}$ is the field strength in the non-commutative geometry defined in [6]. Using the vierbein (5.3), we see that the equation (5.10) is equivalent to

$$\mathrm{Tr}(J_G \hat{F}) = 0\,. \tag{5.12}$$

Therefore, recalling that B is of type $(1,1)$, we arrive at the non-commutative hermitean Yang-Mills equations:

$$\begin{aligned} \hat{F}^{2,0} &= 0 \\ \hat{F} \wedge J_G^{\frac{p-1}{2}} &= 0\,. \end{aligned} \tag{5.13}$$

It should be stressed that the formulae mapping to noncommutative Yang-Mills theory used above apply to constant fieldstrengths F of rank one, and moreover to backgrounds with constant B. Nevertheless, our BPS conditions apply to nonconstant B and F, and admit natural nonabelian generalizations, so it would be nice to establish the equivalence between (5.9b) and (5.13) in the more general setting. In particular, it would be interesting to see if there is still a map to the noncommutative geometry defined using the $*$ product of [23].

6. Relations to previously studied nonlinear deformations of the instanton equations

The nonlinear deformations of instanton equations we have found from κ-symmetry and BPS conditions are closely related to some nonlinear instanton equations which have been previously studied.

First, the equation (4.15) has some similarities to the nonlinear deformations of the hermitean Yang-Mills equations studied in [8]. Leung introduced his equations to study the relation of Gieseker and Mumford stability of holomorphic vector bundles. Among Leung's results are some results that suggest that there should be a 1-1 correspondence between the solutions of the deformed and undeformed equations. This is certainly consistent with the change of variables discussed by Seiberg and Witten. Indeed, it suggests that their change of variables might be useful in studying stability of holomorphic vector bundles.

Second, it is worth pointing out that the equations (5.8) together with $F^{2,0} = 0$ are just the equations of motion of the "chiral cocycle theories" studied by Losev et. al. in [9]. These are theories of type $(1,1)$ connections on holomorphic bundles governed by actions formed from Bott-Chern classes. The chiral cocycle lagrangeans $\mathcal{L}_n$ exist for complex manifolds X_n of any complex dimension n. They are constructed using Bott-Chern forms and are functionals of a gauge field A satisfying $F^{0,2} = 0$. They have equation of motion:

$$\delta \int_{X_n} \mathcal{L}_n[g] = (\bar{\partial}(g^{-1}\partial g))^n = 0\,, \tag{6.1}$$

where $A^{0,1} = -\bar{\partial} g g^{-1}$ and $g \in \mathrm{GL}(N, \mathbb{C})$. Therefore, using the lagrangean

$$\sum_{k=0}^{n} a_k \int_{X_n} J^{n-k} \mathcal{L}_k[g] \tag{6.2}$$

for suitable coefficients a_k we can reproduce equations (4.15) above and their limits (5.7) and (5.9b). This connection is potentially useful because, as discussed at length in [9], the theories are partially solvable using higher dimensional current algebra and higher dimensional analogues of the "bc-systems" of 2D conformal field theory. One wonders if the higher-dimensional fermionization described in [9] could be useful in this regime of string theory.

Moreover, the equations of [8] and of [9] both admit natural *nonabelian* generalizations. The correct formulation of a nonabelian Born-Infeld theory is a problem which has been partially, but not fully solved [24]. One also wonders if the nonabelian chiral cocycle equations will be the equations for BPS configurations of nonabelian Born-Infeld theories. If this is the case then the connection could be very rich for mathematical physics, providing natural nonlinear deformations of Yang-Mills-Higgs systems.

7. Kodaira-Spencer theory and the M5-brane

The above analysis can also be applied to the κ-symmetries of the $M2$ and $M5$ branes. In the case of the $M2$ brane, the analysis has already been done in [1]. Since the only worldvolume fields are scalars there is no nontrivial rotation of the Γ operator. This is consistent with the fact that we found no interesting deformations for the case of $D1$ and $D2$ branes.

The situation for the M5 brane, on the other hand, is much more nontrivial. Supersymmetric configurations on the M5 brane have been studied in [25, 26, 27]. Here we focus on M5 instantons with none of the 5 normal bundle scalars activated. Since we are working with instantons we must decide on a formulation of the 5brane theory, as well as a continuation of that theory to euclidean space. Since we are interested in on-shell configurations we restrict attention to the purely on-shell and covariant formulation of [10][12]. In this theory one uses a self-dual 3-form $h_{\mu\nu\rho}$ nonlinearly related to the field strength of the 2-form potential, $H_{\mu\nu\rho}$. The latter fieldstrength satisfies the Bianchi identity $dH \propto f^*(G_4)$ where G_4 is the M-theory 4-form fieldstrength.

In the formulation of [12] one begins (in Minkowski space) with a real self-dual 3-form $*h = h$. The nonlinear equation of motion for h is

$$\begin{aligned} \mathcal{M}^{\mu\nu} \partial_\mu h_{\nu\lambda\rho} &= 0 \\ \mathcal{M}_\mu{}^\nu &= \delta_\mu{}^\nu - 2 h_{\mu\rho\lambda} h^{\nu\rho\lambda} \,. \end{aligned} \tag{7.1}$$

The Γ operator defining κ-symmetry transformations is simply given by

$$\Gamma = \Gamma_{(0)} \left(1 - \frac{1}{2 \cdot 3!} h_{\mu\nu\rho} \gamma^{\mu\nu\rho}\right).$$

The crucial nonlinear relation of h to the fieldstrength H of the 2-form potential is, according to [12], given by

$$H_{\mu\nu\rho} = \left(\mathcal{M}^{-1}\right)_\mu{}^\lambda h_{\lambda\nu\rho}\,. \tag{7.2}$$

While H satisfies a simple Bianchi identity $dH = 0$ (in a background with $G_4 = 0$) the self-duality condition and the Γ operator are complicated nonlinear functions of H. Indeed, we will regard the relation of h to H as quite analogous to that between Y and M, explored extensively in the previous sections. In particular, we have learned from our previous results that, while the equations for supersymmetric brane configurations are complicated nonlinear equations on M, they become much simpler in terms of Y. An analogous phenomenon proves to be the case in the M5 theory.

Accordingly, let us examine the conditions on h for a supersymmetric M5 instanton. We will continue to euclidean space by relaxing the reality condition on h and taking

$$*h = -ih\,. \tag{7.3}$$

On a Kähler manifold this implies that h is of the form $h = h^{3,0} + h^{2,1} + h^{1,2}$ where $h^{2,1}$ is in the image of $J\wedge$ and $h^{1,2}$ is in the kernel of $J\wedge$. We take the equation of motion on a curved euclidean manifold X to be:

$$\mathcal{M}^{\mu\nu} \nabla_\mu h_{\nu\lambda\rho} = 0\,. \tag{7.4}$$

This implies $dH = 0$ [12]. Finally, we can continue Γ to euclidean space by taking

$$\Gamma = \mp i \Gamma_{(0)} \left(1 - \frac{1}{2 \cdot 3!} h_{\mu\nu\rho} \gamma^{\mu\nu\rho}\right). \tag{7.5}$$

The condition (7.3) on h guarantees that $\Gamma^2 = 1$.

We now take X to be a Calabi-Yau 3-fold and look for on-shell field configurations h such that there are covariantly constant spinors with $\Gamma\eta = \eta$. Choosing the lower sign in (7.5) we find no condition on h. Choosing the upper sign we find the general solution

$$h = c\Omega + \chi^{1,2} \tag{7.6}$$

where c is a constant, $\chi^{1,2}$ is of type $(1,2)$, and $J \wedge \chi^{1,2} = 0$, or equivalently $g^{m\bar{n}} \chi^{1,2}_{m\bar{n}\bar{p}} = 0$. It is now useful to define the variable

$$\mu_m{}^{\bar{n}} := \frac{1}{2} \Omega_{mpq} \chi^{\bar{n}pq}. \tag{7.7}$$

The condition $J \wedge \chi^{1,2} = 0$ implies

$$\mu_{mn} = \mu_{nm}. \tag{7.8}$$

Now we examine the implications of (7.4). This equation has $(2,0)$, $(1,1)$, and $(0,2)$ components. We find that the $(2,0)$ component is identically satisfied thanks to $\nabla\Omega = 0$. The $(0,2)$ component becomes

$$g^{m\bar{n}}\nabla_{\bar{n}}\chi_{m\bar{p}\bar{q}} - 4\chi^{n}{}_{r\bar{s}}\chi^{mr\bar{s}}\nabla_{n}\chi_{m\bar{p}\bar{q}} = 0 \tag{7.9}$$

which is a deformation of the standard gauge fixing condition $\partial^{\dagger}\chi^{1,2} = 0$ of Kodaira-Spencer theory, where $\partial^{\dagger} : \Omega^{1,2} \to \Omega^{0,2}$. Finally, using (7.8) repeatedly the $(1,1)$ component of (7.4) becomes *precisely* the Kodaira-Spencer equation

$$\partial_{[m}\mu_{n]}{}^{\bar{p}} - 8c\mu_{[m}{}^{\bar{q}}\partial_{\bar{q}}\mu_{n]}{}^{\bar{p}} = 0\,, \tag{7.10}$$

for a finite deformation μ of the complex structure on X, as long as $c \neq 0$. The problem we face at this point is that our three equations for μ (or χ), (7.8), (7.9) and (7.10), are potentially overdetermined, hence it is not clear that they have solutions. We conjecture that solutions in fact do exist, and that on a Calabi-Yau manifold they are in one to one correspondence with the solutions to the standard Kodaira-Spencer equations. We will now give some partial evidence for this. The Kodaira-Spencer equation has been explicitly solved on a Calabi-Yau manifold by Tian and Todorov in [28, 29]. The first step in doing this is to set up a perturbative procedure to solve the equation. We start from the ansatz:

$$\mu = \sum_{n=1}^{\infty} \epsilon^{n}\mu^{(n)}\,, \tag{7.11}$$

where ϵ is a formal parameter. We will denote by $' : \Omega^{(p,0)}(\wedge^{q}\overline{T}_{X}) \to \Omega^{(p,3-q)}$ the contraction with $\overline{\Omega}$, so that $\chi = \mu'$. Making a convenient choice of c, the Kodaira-Spencer equation at n^{th} order is given by:

$$\partial\mu^{(n)} + \frac{1}{2}\sum_{i=1}^{n}[\mu^{(i)}, \mu^{(n-i)}] = 0\,. \tag{7.12}$$

The resulting equations can be recursively solved in the gauge $\partial^{\dagger}\chi = 0$. We will denote a solution in this gauge by χ_{TT}. At first order, one finds that $\chi^{(1)}_{TT}$ is harmonic. At second order, and using that $[A,B]' = \overline{\partial}(A \wedge B)'$, the solution is given by:

$$\chi^{(2)}_{TT} = -\partial^{\dagger}\frac{1}{2\Delta_{\partial}}\overline{\partial}(\mu^{(1)}_{TT} \wedge \mu^{(1)}_{TT})'\,.$$

One can in fact find an explicit solution χ_{TT}, constructed in a recursive way, which satisfies the gauge condition $\partial^{\dagger}\chi_{TT} = 0$ and also $\overline{\partial}\chi_{TT} = 0$. This solution is given, at nth order, by [28, 29]:

$$\chi^{(n)}_{TT} = -\partial^\dagger \frac{1}{2\Delta_\partial} \sum_{i=1}^{n} \overline{\partial}(\mu^{(i)}_{TT} \wedge \mu^{(n-i)}_{TT})' \,, \tag{7.13}$$

and $\chi^{(1)}_{TT}$ is any harmonic $(1,2)$ form in the Calabi-Yau. A remarkable fact is that this solution satisfies automatically the extra equation (7.8), or equivalently, $J\wedge\chi_{TT} = 0$. This can be proved inductively as follows: Consider the $(2,3)$-form $J \wedge \chi^{(1)}_{TT}$. Since $\chi^{(1)}_{TT}$ is harmonic, using the Hodge identity $[J\wedge, \partial^\dagger] = i\overline{\partial}$ we can easily prove that this $(2,3)$-form is also harmonic. But $h^{2,3} = 0$ on a Calabi-Yau, so $J \wedge \chi^{(1)}_{TT} = 0$. This proves (7.8) at first order. Let's now assume that $J \wedge \chi^{(i)}_{TT} = 0$ for $i = 1, \ldots, n-1$. Then

$$J \wedge \partial^\dagger \frac{1}{2\Delta_\partial} \overline{\partial}(\mu^{(i)}_{TT} \wedge \mu^{(n-i)}_{TT})' = \partial^\dagger \frac{1}{2\Delta_\partial} \overline{\partial}\Big(J \wedge (\mu^{(i)}_{TT} \wedge \mu^{(n-i)}_{TT})'\Big) \,, \tag{7.14}$$

and $J \wedge (\mu^{(i)}_{TT} \wedge \mu^{(n-i)}_{TT})'$ is easily seen to be zero after using the definition of $'$ and the induction hypothesis. Our equations involve the deformed gauge condition (7.9) rather than the one used in the proof of the Tian-Todorov theorem. Since we are just changing the gauge, we can try to find a solution to our equations of the form $\chi^{(n)} = \chi^{(n)}_{TT} + \delta\chi^{(n)}$, where $\chi^{(n)}_{TT}$ is the explicit solution (7.13), and in such a way that (7.8) is still true. The first step in doing this is to rewrite (7.9) as follows. We introduce the determinant of μ, that we will denote by $\det \mu := \det_{i,\bar{j}} \mu_i{}^{\bar{j}}$. If we choose $||\Omega||^2 = 1$, we find that (7.9) can be written as

$$\nabla^m \mu_{mk} = -\frac{1}{8c}(\nabla_k - 8c\mu_{km}\nabla^m) \log[1 + 64c(\det \mu)] \,. \tag{7.15}$$

We can also write (7.15) as

$$(\partial^\dagger \chi)_{\bar{p}\bar{q}} = (\overline{\partial}^\dagger (f\overline{\Omega}))_{\bar{p}\bar{q}} - 8c\chi_{m\bar{p}\bar{q}}\nabla^m f \,, \tag{7.16}$$

where $f = -\frac{1}{8c}\log[1 + 64c(\det \mu)]$. At first and second order, the solution to our equations is just given by $\chi^{(n)}_{TT}$, $n = 1, 2$, since the deformation of the gauge fixing condition is cubic in χ. At third order, the gauge fixing condition becomes:

$$(\partial^\dagger \chi^{(3)})_{\bar{p}\bar{q}} = (\partial^\dagger \delta\chi^{(3)})_{\bar{p}\bar{q}} = (\overline{\partial}^\dagger (f^{(3)}\overline{\Omega}))_{\bar{p}\bar{q}} \,, \tag{7.17}$$

where $f^{(3)}$ is the third order term in μ (and in fact involves only $\mu^{(1)}$). A change of gauge at third order simply means that $\delta\chi^{(3)} = \partial\nu$, where ν is a $(0,2)$ form that satisfies:

$$\Delta_\partial \nu = \overline{\partial}^\dagger (f^{(3)}\overline{\Omega}) \,. \tag{7.18}$$

We can then write

$$\delta\chi^{(3)} = \partial \frac{1}{\Delta_\partial} \overline{\partial}^\dagger (f^{(3)}\overline{\Omega}) \,, \tag{7.19}$$

and it is easy to check (using again the Hodge identities) that $J \wedge \delta\chi^{(3)} = 0$. Therefore, the perturbation of the solution (7.13) induced by the deformation of the gauge condition preserves (7.8) at third order. Unfortunately, the procedure becomes cumbersome for higher orders and we have not been able to check it for the next terms in the perturbative series. We conjecture, however, that the equations with the new gauge fixing condition can be solved in the way that we have sketched. (In arranging a full proof it might help to notice that the right hand side of (7.15) involves the deformed holomorphic derivative $\nabla_k - 8c\mu_{km}\nabla^m$.) In particular, we conjecture that the solutions to our equations are in one to one correspondence with the solutions to the Kodaira-Spencer equation with the usual gauge fixing, and in such a way that (7.8) is satisfied. Our result is relevant to the problem of computing nonperturbative corrections for M-theory on a Calabi-Yau. These will involve a weighted sum over all configurations of wrapped fivebranes with supersymmetric H fields turned on. The preceding relates such H fields to points in the moduli space of complex structures on the Calabi-Yau. H is subject to a quantization condition which restricts the sum to rational points reminiscent of those arising from the attractor equations [30, 31].

Our result also establishes a very direct relation between the M5-brane and Kodaira-Spencer theory. Connections between the M5 theory and Kodaira-Spencer theory have been discussed before. In particular, in [32] Witten related the quantization of the phase space $H^3(X)$ to Kodaira-Spencer theory and the holomorphic anomaly equation [33]. (See [36] for some recent progress.) In [34] Witten then connected the quantization of $H^3(X)$ to the M5 theory. Closely related connections have been explored in [35]. Nevertheless, we believe the above connection is new. We hope it leads to further progress in demystifying the M-theory fivebrane.

Acknowledgments

We would like to thank D. Gross, N. Seiberg, G. Tian and S.-T. Yau for useful discussions on noncommutative geometry and deformations of instantons. G.M. and A.S. would like to acknowledge the hospitality of the Aspen Center for Physics. M.M. would like to thank L. Álvarez-Cónsul and O. Garcia-Prada for pointing out reference [8]. G.M. would like to thank R. Dijkgraaf, A. Gerasimov, A. Losev and S. Shatashvili for discussions (c. 1997) of relations between fivebranes and Kodaira-Spencer theory. This work was supported by DOE grants DE-FG02-92ER40704 and DE-FG02-91ER40654.

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